SPECTRAL SEQUENCES

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1. Introduction

Let $F : \mathcal{A} \to \mathcal{B}$ and suppose $\mathcal{A}$ has enough injective objects so that F admits a right derived functor. Define the *cohomological dimension of* F to be

$$\operatorname{cd} F := \max\{\sup\{n \in \mathbb{Z}_{\geq 0} : R^n F(A) \neq 0 \text{ for some } A \in \mathcal{A}\}, 0\}.$$

Theorem 1.1. *Let*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be a diagram of left exact functors between abelian categories, and suppose $\mathcal{A}$ and $\mathcal{B}$ admit enough injectives. Suppose further[1] *that F sends injective objects to G-acyclic objects. Then*

$$\operatorname{cd}(G \circ F) \leq \operatorname{cd} G + \operatorname{cd} F.$$

This follows from

Proposition 1.2. *Suppose the same hypotheses as in theorem 1.1.*

For each $n \in \mathbb{Z}_{\geq 0}$ and $A \in \mathcal{A}$, there is a subobject series of $R^n(G \circ F)(A)$ consisting of factors which are subquotients of $R^q G(R^p F(A))$ for $p + q = n$.

This will follow *by definition* from the statement

'there is a convergent E_2-spectral sequence $R^p G(R^q F(A)) \Rightarrow R^{p+q}(G \circ F)(A)$'

2. Filtration on a chain complex

We fix an abelian category $\mathcal{A}$. We first state our main theorem without mentioning spectral sequences by name:

Theorem 2.1. *Let $C^\bullet$ be a chain complex concentrated in nonnegative degree, and suppose we have a filtration $\{F^m C^\bullet\}_{m \leq 1}$ on $C^\bullet$ such that for each $n \geq 0$*

$$0 = F^1 C^n \subset F^0 C^n \subset \cdots \subset F^{-n+1} C^n \subset F^{-n} C^n = C^n.$$

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[1]This hypothesis is satisfied if F is the left adjoint of some functor $\mathcal{B} \to \mathcal{A}$, in which case F sends injective objects to injective objects.

Then there is a collection of objects $\{E_r^{pq}\}_{r\geq 0,p,q}$ *concentrated in* $p, q \geq 0$ *and morphisms* $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ *such that the obvious connected diagrams are chain complexes and* E_{r+1}^{pq} *is isomorphic the cohomology at* d_r^{pq} *with the properties such that*

- *for each* $p, q \geq 0$, $E_0^{pq} = F^pC^q/F^{p+1}C^q$*;*
- *for each* $n \geq 0$*, the stabilising objects* $\{E_\infty^{p,n-p}\}_{0\leq p\leq n}$ *of the sequences* $\{\{E_r^{p,n-p}\}_r\}_{0\leq p\leq n}$ *appear as the factors in the induced filtration*

$$\{F^pH^n\}_{p\leq 1} \qquad \textit{where} \qquad F^pH^n := \mathrm{Im}(H^n(F^pC^\bullet) \to H^n(C^\bullet))$$

on $H^n(C^\bullet)$*:*

$$\begin{tikzcd}[row sep=small]
F^{-n}H^n & = H^n(C^\bullet) \\
F^{-n+1}H^n \arrow[u, hook, "E_\infty^{n,0}"] & \\
F^{-n+2}H^n \arrow[u, hook, "E_\infty^{n-1,}"] & \\
\vdots \arrow[u, hook] & \\
F^0H^n \arrow[u, hook] & \\
F^1H^n \arrow[u, hook, "E_\infty^{0,n}"] & = 0
\end{tikzcd}$$

Following Weibel we call a filtration of the type occuring in the hypothesis of the theorem *canonically bounded*. Abstractly, we call a collection of objects $\{E_r^{pq}\}_{r\geq 0,p,q}$ and morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ appearing in the above theorem a *first quadrant* E_0*-spectral sequence*

The subdiagram of a spectral sequence consisting of objects and morphisms with a fixed $r \geq 0$ is called the r*th page.*

First quadrant E_0-spectral sequences (of objects of $\mathcal{A}$) form a category, where we define a morphism in an obvious way.

Lemma 2.2 (Mapping lemma)**.** *Let* $f : \{E_r^{pq}\} \to \{E_r'^{pq}\}$ *be a morphism of first quadrant* E_0*-spectral sequences. If* f *induces an isomorphism of diagrams on the* r*th page, then* f *induces an isomorphism of diagrams on the* s*th page whenenever* $s \geq r$*.*

Proof. Use the 5-lemma. $\square$

For a first quadrant E_0-spectral sequence $\{E_r^{pq}\}_{r\geq 0,p,q}$, and a family $\{A^n\}_{n\geq 0}$ of objects, we define:

$$E_0^{pq} \Rightarrow A^{p+q}$$

to mean that for each $n \geq 0$, there is a descending filtration of A^n with factors $E_\infty^{n,0}$, $E_\infty^{n-1,1}$, $\ldots$, $E_\infty^{0,n}$ So we could have restated our theorem as follows:

Theorem 2.3. *Let $C^\bullet$ be a chain complex concentrated in nonnegative degree and suppose we have a canonically bounded filtration $\{F^nC^\bullet\}_{n\geq 0}$ on $C^\bullet$. Then there is a first quadrant E_0-spectral sequence such that*

$$F^pC^q/F^{p-1}C^q \Rightarrow H^n(C^\bullet)$$

Exercise 2.4. Prove theorem 2.1 using Weibel.
[Hint: it may be easier to prove the version for homological spectral sequences to avoid reindexing; strictly speaking this is not the same as theorem 2.1; why?]

3. Applications

Let $B^{\bullet\bullet}$ be a first quadrant bicomplex. We obtain two canonically bounded filtrations of $C^\bullet := \operatorname{Tot} B^{\bullet\bullet}$

$${}_IF^{-1}: \quad 0^\bullet$$

$${}_IF^0: \quad 0 \longrightarrow B^{00} \longrightarrow B^{01} \longrightarrow B^{02} \longrightarrow \cdots$$

$${}_IF^1: \quad 0 \longrightarrow B^{00} \longrightarrow B^{01} \oplus B^{10} \longrightarrow B^{02} \oplus B^{11} \longrightarrow \cdots$$

$${}_IF^2: \quad 0 \longrightarrow B^{00} \longrightarrow B^{01} \oplus B^{10} \longrightarrow B^{02} \oplus B^{11} \oplus B^{20} \longrightarrow \cdots$$

$$\vdots$$

and

$$
\begin{array}{ll}
{}_{II}F^{-1}: & 0^\bullet \\[2em]
{}_{II}F^{0}: & 0 \longrightarrow B^{00} \longrightarrow B^{10} \longrightarrow B^{20} \longrightarrow \cdots \\[2em]
{}_{II}F^{1}: & 0 \longrightarrow B^{00} \longrightarrow B^{01} \oplus B^{10} \longrightarrow B^{11} \oplus B^{20} \longrightarrow \cdots \\[2em]
{}_{II}F^{2}: & 0 \longrightarrow B^{00} \longrightarrow B^{01} \oplus B^{10} \longrightarrow B^{02} \oplus B^{11} \oplus B^{20} \longrightarrow \cdots \\[2em]
\vdots &
\end{array}
$$

and note that we have

$$ {}_{I}E_0^{pq} ={}_{I} F^pC^p/{}_{I}F^{p-1}C^p = B^{pq} ={}_{II} F^pC^p/{}_{II}F^{p-1}C^p ={}_{II} E_0^{pq}. $$

Proposition 3.1 (Cartan–Eilenberg resolutions)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories and suppose $\mathcal{A}$ has enough injectives. Let $X^\bullet \in \mathrm{Ch}^+(\mathcal{A})$. Then there exists a first quadrant bicomplex $J^{\bullet\bullet}$ of injective objects equipped with a morphism of chain complexes $X^\bullet \to J^{\bullet,0}$ which induces injective resolutions $X^p \to J^{p,\bullet}$ for each $p \geq 0$, such that the induced maps on cohomology $H^p(X^\bullet) \to H^p_{\text{horisontal}}(J^{\bullet\bullet})$ induce injective resolutions.*

Proof. Use the Horseshoe lemma and splice. □

Corollary 3.2. *Suppose the same hypotheses as in theorem 1.1.*

There exists a first quadrant E_0-spectral sequence $E_0^{pq} \Rightarrow R^{p+q}(G \circ F)(A)$ whose 2nd page is given by $E_2^{pq} = R^qG(R^pF(A))$.

Proof. Take an injective resolution $A \to I^\bullet$ and by proposition 3.1 let $J^{\bullet\bullet}$ be a Cartan–Eilenberg resolution of the G-acyclic complex $F(I^\bullet)$.

We now consider the two spectral sequences associated to the bicomplex $B^{\bullet\bullet} := G(J^{\bullet\bullet})$.

$\boxed{I}$ Consider the first quadrant E_0-spectral sequence ${}_{I}E_0^{pq}$.

The columns of $B^{\bullet\bullet}$ are truncations of G applied to a bounded below exact sequence of G-acyclic objects:

$$
\begin{array}{ccccc}
\begin{array}{c}\text{exact, } G\text{-acyclic objects,}\\ \text{and bounded below}\end{array} & & \text{exact} & & \begin{array}{c}\text{exact everywhere except}\\ H^0(B^{p\bullet}) = GFI^p\end{array} \\[2em]
\begin{array}{c} \vdots \\ \uparrow \\ J^{p1} \\ \uparrow \\ J^{p0} \\ \uparrow \\ FI^p \\ \uparrow \\ 0 \end{array}
& \begin{array}{c}\text{apply } G\\ \rightsquigarrow\end{array} &
\begin{array}{c} \vdots \\ \uparrow \\ B^{p1} \\ \uparrow \\ B^{p0} \\ \uparrow \\ GFI^p \\ \uparrow \\ 0 \end{array}
& \begin{array}{c}\text{truncate}\\ \rightsquigarrow\end{array} &
\begin{array}{c} \vdots \\ \uparrow \\ B^{p1} \\ \uparrow \\ B^{p0} \\ \uparrow \\ 0 \end{array}
\end{array}
$$

hence the 1st page is concentrated at $p = 0$ with terms $GFI^\bullet$, which by definition is $G \circ F$ applied to a truncated injective resolution of A. It is clear that this spectral sequence stabilises by the 1st page with

$$
{}_I E_0^{pq} \Rightarrow R^{p+q}(G \circ F)(A).
$$

$\boxed{II}$ Now consider the first quadrant E_0-spectral sequence ${}_{II}E_0^{pq}$.

Consider the extended bicomplex of $J^{\bullet\bullet}$ with the morphism $FI^\bullet \to J^{\bullet 0}$

$$
\begin{array}{c} \vdots \\ \uparrow \\ J^{\bullet 1} \\ \uparrow \\ J^{\bullet 0} \\ \uparrow \\ FI^\bullet \\ \uparrow \\ 0 \end{array}
$$

Since each object in this diagram is G-acyclic, and the horisontal chain complexes are bounded below, taking cohomology commutes with respect to applying G; this gives us isomorphisms of chain complexes for each $p \geq 0$:

$$\begin{tikzcd}
\vdots & \vdots \\
H^p(B^{\bullet 1}) \arrow[u] \arrow[r, "\sim"] & GH^p(J^{\bullet 1}) \arrow[u] \\
H^p(B^{\bullet 0}) \arrow[u] \arrow[r, "\sim"] & GH^p(J^{\bullet 0}) \arrow[u] \\
H^p(GFI^{\bullet}) \arrow[u] \arrow[r, "\sim"] & GH^p(FI^{\bullet}) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

It follows that the 2nd page of the spectral sequence is given by

$$E_2^{pq} = R^q G(R^p F(A)).$$

□

The statement of the previous theorem is a bit clunky. In general, if we have first quadrant E_0-spectral sequence, $\{E_r^{pq}\}_{r\geq 0,p,q}$, we can consider the subdiagram $\{E_r^{pq}\}_{r\geq a,p,q}$ for any $a \geq 0$, which we call a first quadrant E_a-spectral sequence. We then write

$$E_a^{pq} \Rightarrow A^{p+q}$$

to mean there exists a first quadrant E_0-spectral sequence with the same ath page (with the same ath differentials) as $\{E_a^{pq}\}_{p,q}$ such that $E_0^{pq} \Rightarrow A^{p+q}$. This way we have defined the symbol $\Rightarrow$ for each page of a first quadrant E_0-spectral sequence.

We can restate the previous theorem:

Theorem 3.3 (Grothendieck spectral sequence)**.** *Suppose the same hypotheses as in theorem 1.1.*

There is a first quadrant E_2-spectral sequence

$$R^q G(R^p F(A)) \Rightarrow R^{p+q}(G \circ F)(A).$$

Obviously it is possible to define first quadrant E_a-spectral sequences with $a \geq 0$ independently of E_0-spectral sequences, but for our purposes this wasn't necessary.